November 16, 1972

Dear Bill,

Thanks for your preprints, which came at a very opportune time. I was about to assume in my Antwerp notes that $p \neq 2$, because I couldn't prove the following facts, which is of course a consequence of "The restriction…"

The support of the character of an absolutely cuspidal representation of $\mathrm{GL}(2,F)$, *F a non-archimedean local field, does not contain* $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$ *if* $|\alpha| \neq |\beta|$.

As an aside, what does the word p-adic mean to you? Can a p-adic field have positive characteristic? I knew of course that this assertion should be a consequence of a Shalika-Mautner type theorem. But my attempts at finding a statement like your Theorem 3 failed.

Just to make sure that I am using your results correctly let me sketch the verification of the above corollary. σ : the representation of $U = Z(F)G(O_F)$ which you denote ϵP. $\Pi = \mathrm{Ind}\big(\sigma, G(F,u)\big)$. Commuting algebra of Π formed by functions λ such that

$$(*) \qquad \lambda(k_1 h k_2) = \sigma(k_2^{-1})\lambda(h)\sigma(k_1^{-1}).$$

If φ such that $\varphi(ug) = \sigma(u)\varphi(g)$, $u \in U$, then $\lambda : \varphi \to \psi$ with

$$\psi(h) = \int_{Z(F)\backslash G(F)} \lambda(g)\varphi(gh)$$

Consider a λ satisfying $(*)$ and take

$$h = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \qquad |a| \leqslant |b|$$

ϖ is a generator of the maximal ideal of O_F. From $(*)$

$$\sigma(k^{-1})\lambda(h) = \lambda(h)\sigma(hkh^{-1})$$

if

$$k \in U \cap h^{-1}Uh.$$

In particular if $x \in O_F$

$$\sigma\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\lambda(h) = \lambda(h)\sigma\begin{pmatrix} 1 & \frac{ax}{b} \\ 0 & 1 \end{pmatrix}$$

It follows from your analysis that if your c is even then $\lambda(h)$ is 0 unless $|a| = |b|$ and that if c is odd $\lambda(h)$ is 0 unless $|a| = |b|$ or $\left|\frac{a}{b}\right| = |\varpi|$. Moreover in the second case, $\mathrm{trace}\,\lambda(h) = 0$ when $\left|\frac{a}{b}\right| = |\varpi|$. More generally if h has eigenvalues in F, there is a k in $G(O_F)$ so that

$$k^{-1}hk = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} = g \qquad \left|\frac{a}{b}\right| \leqslant 1.$$

If $|a| < |b|$

$$\mathrm{trace}\,\lambda(h) = \mathrm{trace}\,\sigma(k^{-1})\lambda(h)\sigma(k) = \mathrm{trace}\,\lambda(g) = 0.$$

Let $\{g_\alpha\}$ be a set of representations for $U\backslash G(F)$. Let σ act on V and Π on W. If $\{v_\beta\}$ is a basis of V, we may take as a basis for W the functions $\varphi_{\alpha\beta}$, where $\varphi_{\alpha\beta}$ is 0 outside of Ug_α and $\varphi_{\alpha\beta}(ug_\alpha) = \sigma(u)v_\beta$. We take g_α of the form $t_m k$ where

$$t_m = \begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} \qquad m \geqslant 0$$

and where k runs over a set of representatives for $K \cap t_m^{-1} K t_m \backslash K$. The $\varphi_{\alpha\beta}$ can be used to calculate the trace.

What we have to do to prove the corollary is to show that if

$$h_0 = \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} \qquad \left| \frac{a_0}{b_0} \right| \neq 1$$

and if t is a function with support in a small neighbourhood of h_0 then

$$\operatorname{trace} \lambda\Pi(f) = 0$$

for any λ in the commuting algebra.

We may suppose that if $f(h) \neq 0$ then h has eigenvalues in F of different absolute values. Let W_α be the functions with support in Kg_α. Let $\lambda\Pi(f) = A$ and let $A = (A_{\alpha\beta})$ where $A_{\alpha\beta} : W_\alpha \to W_\beta$. We have to show that trace $A_{\alpha\alpha} = 0$. Now $A : \varphi \to \psi$ with

$$\psi(x) = \int_{G(F)} \lambda(g)\varphi(gxh)\, dg$$

$A_{\alpha\alpha} : \varphi \to \psi$ with

$$\begin{aligned} \psi(g_\alpha) &= \int_{\{\, g \mid gg_\alpha h = ug_\alpha,\ u \in U \,\}} \lambda(g)\varphi(gg_\alpha h)\, dg \\ &= \int_{\{\, g \mid gg_\alpha h = ug_\alpha \,\}} \lambda(g)\sigma(u)\varphi(g_\alpha)\, dg \end{aligned}$$

Thus

$$A_{\alpha\alpha} = \int_{\{\, g \mid gg_\alpha h = ug_\alpha \,\}} \lambda(g)\sigma(u)\, dg.$$

Since

$$\lambda(g)\sigma(u) = \lambda(u^{-1}g) = \lambda(g_\alpha h^{-1} g_\alpha^{-1})$$

we have

$$\operatorname{trace} A_{\alpha\alpha} = \int \operatorname{trace} \lambda(g_\alpha h^{-1} g_\alpha^{-1}) = 0.$$

Thanks once again for the preprints,
Bob

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