Notation

$\widehat{G^0}$: associate group to G.
$\widehat{G^0_{\rm ad}}$: associate group to $G_{\rm ad}$. $\widehat{G^0_{\rm ad}}$ is simply-connected.
$\widehat{G^0_{\rm sc}}$: associate group to $G_{\rm sc}$. $\widehat{G^0_{\rm sc}}$ is adjoint.

$\widehat{T^0} \subseteq \widehat{B^0}$: CSG in a BSG of $\widehat{G^0}$.
$\widehat{T^0_{\rm ad}}$: Cartan subgroup of $\widehat{G^0_{\rm ad}}$ corresponding to $\widehat{T}$.
$\widehat{T^0_{\rm sc}}$: Cartan subgroup of $\widehat{G^0_{\rm sc}}$ corresponding to $\widehat{T}$.

$\widehat{L^0}$: lattice of rational characters of $\widehat{T^0}$.
$\widehat{L}_{\rm ad}$: lattice of rational characters of $\widehat{T^0_{\rm ad}}$.
$\widehat{L}_{\rm sc}$: lattice of rational characters of $\widehat{T^0_{\rm sc}}$.

$\widehat{G}$: the associate group in the form $\widehat{G^0} \times \mathbf{Z}$. Thus we are only interested in unramified phenomena.

F: non-archimedean local field.

G: always quasi-split over F and split over an unramified extension.

$T^0 \subseteq B$: CSG and BSG over F.

[2]

Intertwining operators and reducibility

In this section I am going to manipulate some facts about intertwining operators. I have not unified them myself. They are just facts which either you or H.-C. have mentioned to me and which I am willing to take for granted. I will then draw some consequences from them.

An element of the unitary principal series is determined by a $t \in \widehat{T^0} \times \mathbf{Z}$ which projects to $1 \in \mathbf{Z}$. Only the conjugacy class of t matters.

The intertwining operators are formally

$$\varphi \to \varphi'(g) = \int_{N(F)\cap w^{-1}N(F)w\backslash N(F)} \varphi(wng)\,dn$$

Here w is an element in the normalizer W of $T(F)$ in $G(F)$. I believe $T(F)\backslash w = \Omega$ is the centralizer of $\mathfrak{G}(\overline{F}/F)$[1] in the Weyl group of T.

[3] We form a special maximum compact K and choose w in K. Then $M(w)$ depends only on ω the image of w in Ω. It takes the function stabilized by K with value 2 at 1 to

$$\frac{\det\bigl(I - |\varpi|A\bigr)}{\det(I - A)}$$

if A is the restriction of the adjoint action of $\operatorname{Ad} w$ to the quotient of $\widehat{\mathbf{n}}$ by $\widehat{\mathbf{n}} \cap \operatorname{Ad} w(\widehat{\mathbf{n}})$.

$M(\omega) = M(w)$ will take $\pi = \pi(t)$ to itself if only if $\omega^{-1}(t) = t$. Recall that Ω acts on T and on $\widehat{T^0}$ or $\widehat{T^0} \times \mathbf{Z} \subseteq \widehat{G}$. Also $M(\omega)$ will be defined at π if and only if $\operatorname{Ad} t$ does not have the eigenvalue 1 on $\widehat{n} \cap \operatorname{Ad} w(\widehat{\overline{n}})$. Let Ω_t be the stabilizer of t in Ω. Let Σ_t be the set of $\widehat{\alpha} > 0$ such that t has the eigenvalue 1 on $\sum_{n\geqslant 0} X_{t^n\widehat{\alpha}}$ (Note $t \in \widehat{T^0} \times \mathbf{Z}$, $t \to 1$ in $\mathbf{Z}$). Let Ω_t^0 be the set of $\omega \in \Omega_t$ which leave Σ_t invariant. **[4]** Ω_t^0 is a group and $M(\omega)$ is defined at π if and only if $\omega \in \Omega_t^0$.

I am assuming that the intertwining operators $M(\omega)$, $\omega \in \Omega_t^0$, span the commuting algebra to $\pi(G)$. If Ω_t^0 is contained in the Levi factor of a PSG of $\widehat{G}$ or of G then all questions can be reduced to this Levi factor. Therefore we may as well assume that Ω_t^0 (or rather its inverse image in W) is contained in a no proper PSG of G over F.

However

$$\sum_{\widehat{\alpha}\in\Sigma_t} \widehat{\alpha}$$

is non-zero if $\Sigma_t \neq \varnothing$ and is fixed by Ω_t^0. We conclude as in the discussion of the basic Lemma that Σ_t must be empty. This means that the connected component of the centralizer of t lies in $\widehat{T^0}$. Let φ be the map $\widehat{G}_{\text{ad}} \to \widehat{G}$ and let $t_{\text{ad}} \in \widehat{G}_{\text{ad}}$ be such that $\varphi(t_{\text{ad}})^{-1}t$ lies in the centre of $\widehat{G^0}$.

I claim that the centralizer of t_{ad} in $\widehat{G^0}$ is the centralizer S of **[5]** $\mathbf{Z}$ in $\widehat{T^0_{\text{ad}}}$. Since there is a basis of $\widehat{L}_{\text{ad}}$ (formed by the fundamental roots of $\widehat{T}_{\text{ad}}$) on which $\mathbf{Z}$ acts by permutations, S is connected and is certainly the connected component of the centralizer. Moreover the centralizer of S is $\widehat{T^0_{\text{ad}}}$. All we have to do is show that the centralizer of t which normalizes S actively centralizes it. If V is the normalizer of S then $T^0_{\text{ad}}\backslash V \simeq \Omega$. I now apply E.4.2 of the *Seminar on Algebraic Groups.* Ω acts on S as a finite reflection group. If $\{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r\}$

[1] Some text has been crossed out before $\mathfrak{G}(\overline{F}/F)$ with "$\mathbf{Z}$" written above it.

is an orbit of $\mathbf{Z}$ in the set of simple roots we form a graph by joining $\widehat{\alpha}_i$, $\widehat{\alpha}_j$ if and only if $\langle\widehat{\alpha}_i, \widehat{\alpha}_j\rangle \neq 0$. Let $O_1, \ldots, O_s$ be the connected components of this graph. As in my Washington Lecture, each O_i has one or two elements and

$$\widehat{\gamma}_i = \sum_{j \in O_i} \widehat{\alpha}_j$$

is a root. **[6]**

$$\omega : \lambda \to \lambda - \sum \langle\lambda, \widehat{\gamma}_i\rangle \gamma_i$$

belongs to Ω. On S, or rather the lattice of one parameter subgroups of S, which may be identified with the invariant elements in $L(T_{\text{ad}})$, it is a reflection in the direction

$$\sum \gamma_i = \sum_{j=1}^{r} \alpha_j$$

because, as one easily checks

$$\gamma_i = \sum_{j \in O_i} \alpha_j$$

The elements $\sum_{j=1}^{r} \alpha_j$ generate the invariants in $L(T_{\text{ad}})$. If $t_{\text{ad}} = s \times 1$ and $\omega(s) = s$ then for any fundamental weight $\widehat{\lambda}$

$$\widehat{\lambda}(s) = \omega\widehat{\lambda}(\omega s) = \omega\widehat{\lambda}(s) = \widehat{\lambda}(s) \prod \widehat{\gamma}_i(s)^{\langle\widehat{\lambda}, \gamma_i\rangle}.$$

However we can choose $\widehat{\lambda}$ so that $\langle\widehat{\lambda}, \gamma_i\rangle = 0$ for all [illegible] i when it is 1. Thus $\widehat{\gamma}_i(s) = 1$ for all [blank]. Since $\widehat{\gamma}_i$ is a root this contradicts the assumption that Σ_t is empty. Thus lemma E.4.2 **[7]** now implies that the centralizer of t_{ad} is S.

The map

$$\omega \to \omega(t_{\text{ad}}) t_{\text{ad}}^{-1}$$

yields therefore an injection of Ω_t^0 into the centre of $\widehat{G}_{\text{ad}}^0$. It is easily seen to be a homomorphism. This shows that Ω_t^0 is abelian.

We wants to apply the basic lemma. To this end we introduce to group N generated by the inverse image of Ω_t^0 in the normalizer of $\widehat{T^0}$ and by t. The splitting of

$$\widehat{T^0} \backslash N \to \mathbf{Z}$$

is to be

$$z \to t^z \equiv 1 \times z \pmod{\widehat{T^0}}$$

Conditions (i) and (v) of the paragraph on the basic lemma are satisfied.

If $t_{\text{ad}} = s \times 1$ we define χ by

$$\chi(\widehat{\lambda}) = \widehat{\lambda}(s) \qquad \widehat{\lambda} \in \widehat{L}_{\text{sc}}$$

[8] Condition (ii) is certainly satisfied and condition (iv) is satisfied because the connected component of the centralizer of s is $\widehat{T^0}$.

We conclude from the basic lemma that the simple factors of G_{ad} must be obtained from a projective linear group over a larger field by restriction of scalars.

This together with the basic lemma is going to allow us to answer the first two questions of the previous letter without any difficulty whatsoever. For the first this is clear. For the second we have only to observe that when $G_{\text{ad}} = \text{PCL}(m)$ then the basic lemma forces

Ω_t^0 to be cyclic of order n and hence t_{ad} to be of the form (up to a permutation and mod similar matrices)

$$\alpha \begin{pmatrix} 1 & & & & \\ & \zeta & & & \\ & & \zeta^2 & & \\ & & & \ddots & \\ & & & & \zeta^{m-1} \end{pmatrix} \qquad \zeta^m = 1$$

It means that the only serious question of the last letter and hence the only serious question of the whole business is the third. You **[9]** will notice that in my suggestion I forgot the denominator.

$$\left|\rho(\gamma)\right| \prod_{\alpha>0} \left|1-\alpha^{-1}(\gamma)\right|$$

[10]

Additional remark

To completely answer the second question one needs to know that the intertwining operators $M(\omega)$, $\omega \in \Omega_t^0$, are linearly independent. Take $G = \mathrm{SL}(m)$. We saw that $M(W)$ multiplies the K invariant vector φ by

$$\frac{\det\bigl(I - |\varpi| A\bigr)}{\det(I - A)}$$

If $\alpha \in T^0_{\mathrm{ad}}(F)$ and $K' = \alpha^{-1} K \alpha$ then

$$\varphi' = \varphi'(y) = \varphi(\alpha g \alpha^{-1})$$

is the K'-invariant vector

$$\begin{aligned}\int_{N(F)\cap w^{-1}N(F)\backslash N(F)} \varphi'(wn)\,dn &= \int \varphi(\alpha w \alpha^{-1} w^{-1} w \alpha n \alpha^{-1})\,dn \\ &= \xi(\beta) \int \varphi(wn)\,dn\end{aligned}$$

Here $\beta \in T^0(F)$ is $\alpha w \alpha^{-1} w^{-1}$. Thus the K'-invariant factor is multiplied by

$$\xi(\beta) \frac{\det\bigl(I - |\varpi| A\bigr)}{\det(I - A)}$$

[11] ξ is of course the character of $T(F)$ defined by t.
If

$$\alpha = \begin{pmatrix} \varpi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and

$$w = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}$$

it is enough to show that the matrix

$$\Bigl(\xi(\alpha^k w^\ell \alpha^{-k} w^{-\ell})\Bigr) \qquad 0 \leqslant k,\ \ell < m$$

is non-singular. This matrix is however

$$(\zeta^{-k\ell}).$$

Its determinant is a Vandermonde determinant which is nonsingular if ζ is a primitive nth root of unity.

[12]

Relation with the study of orbital integrals

From the letters I gave you in Princeton, we know that we have to look for pairs consisting of $v = w \times 1 \in \widehat{G}$ with w in the normalizer of $T^{\widehat{0}}$ and a character χ of $\widehat{L}_{\text{sc}}$ which is trivial on $\left\{ v\lambda - \lambda \;\middle|\; \lambda \in \widehat{L}_{\text{sc}} \right\}$ but different from 1 in all roots $\widehat{\alpha}$.

To χ and v we associated a character η of $\widehat{L}_{\text{sc}}^{\mathfrak{G}(\overline{F}/F)} \backslash \widehat{L}_{\text{ad}}^{\mathfrak{G}(\overline{F}/F)}$. I observe next that if $\widehat{M}$ is a Levi factor of a PSG and $\widehat{M}$ contains both $T^{\widehat{0}}$ and v then η factors through

$$\widehat{L}_{\text{ad}}^{\mathfrak{G}(\overline{F}/F)} \to \widehat{\overline{L}}_{\text{ad}}^{\mathfrak{G}(\overline{F}/F)}$$

if the $^{-}$ denotes the corresponding object for M. According to the few pieces of paper entitled *Scrap* this allows us to assume in addition that v has no nontrivial invariants in $\widehat{L}_{\text{sc}}$.

To justify my observation I remark that if $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r$ are the simple roots of $G^{\widehat{0}}$ and $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_s$ the simple roots of $M^{\widehat{0}}$ and if $\widehat{\mu}_1, \ldots, \widehat{\mu}_r$ are dual to $\alpha_1, \ldots, \alpha_r$, i.e., $\langle \widehat{\mu}_i, \alpha_j \rangle = \delta_{ij}$ **[13]** then the kernel of

$$\widehat{L}_{\text{ad}} \to \widehat{\overline{L}}_{\text{ad}}$$

has basis $\widehat{\mu}_{s+1}, \ldots, \widehat{\mu}_r$ and

$$w\widehat{\mu}_j = \widehat{\mu}_j \qquad j = s+1, \ldots, r.$$

The observation now follows from the definition of η.

Once π has no non-trivial invariants we let N be the group generalized by $T^{\widehat{0}}$ and v and apply the basic lemma. This shows that G_{ad} is the product of groups obtained from projective linear groups by restriction of scalars.

[14]

Scrap

I continue trying to sort out my thoughts. Denote π^g the representation

$$x \to \pi(g^{-1}xg) \qquad g \in G_{\mathrm{ad}}(F)$$

P: parabolic if G over F.
M: Levi factor of P.
τ: irreducible representation of M.
π: normalized induced representation.

Remark. Let M' be the image of M in G_{ad}. There is a map $M' \to M_{\mathrm{ad}}$. Let $m' \in M'(F)$ and let m be its image in $M_{\mathrm{ad}}(F)$. Then $\pi^{m'}$ is induced from τ^m.

Consider the space of functions

$$x \to f(m'^{-1}xm') = f'(x)$$

where f lies in the space of π. If $h \in m$ then

$$f(m'^{-1}hxm') = \delta_P \tau^m(h) f'(x)$$

Thus the action by right translation on this space is induced from τ^m. On the other hand under

$$f \to f'$$

we have

$$\pi^{m'}(g)f(x) = f(xm'^{-1}gm') \to f(m'^{-1}xgm') = f'(xg)$$

On the other hand (from for example 4.13 of Borel-Tits) the map

$$M'(F) \to \operatorname{Im} G(F)\backslash G_{\mathrm{ad}}(F)$$

[15] is surjective.

Therefore if we denote by $C(G)$ the group formerly denoted C we have a surjective map

$$M'(F) \to C(G)$$

However the map

$$M'(F) \to C(M)$$

is also surjective. To see this denote by T^0_G and T^0_M the torus T^0 considered as a subgroup of G and of M respectively. Similarly denote by $\widehat{T}_{\widehat{G}}$, $\widehat{T}_{\widehat{M}}$, $\widehat{T}$ considered as a subgroup of $\widehat{G}$ and $\widehat{M}$.

The diagrams

$$\begin{array}{ccc}
T_{G_{\mathrm{ad}}}(F) & \longrightarrow & \widehat{L}(T_{G_{\mathrm{ad}}}) = L(\widehat{T}_{G_{\mathrm{ad}}}) \\
\downarrow & \searrow & \downarrow \\
G_{\mathrm{ad}}(F) & \longrightarrow & C(G)
\end{array}$$

$$\begin{array}{ccc}
T_{M_{\mathrm{ad}}}(F) & \longrightarrow & \widehat{L}(T_{M_{\mathrm{ad}}}) = L(\widehat{T}_{M_{\mathrm{ad}}}) \\
\downarrow & \searrow & \downarrow \\
M_{\mathrm{ad}}(F) & \longrightarrow & C(M)
\end{array}$$

are commutative and the dotted arrows are surjective.

[16] The map

$$T_{G_{\rm ad}}(F) \to T_{M_{\rm ad}}(F)$$

corresponds to

$$L(\widehat{T}_{G_{\rm ad}}) \to L(\widehat{T}_{M_{\rm ad}})$$

However $\widehat{G}_{\rm ad}$ is simply connected and it is well known (use fundamental highest weights) that this map is surjective.

The character η maybe regarded as a character of $M'(F)$. Suppose it is trivial on the kernel of $M'(F) \to C(M)$. We can factor

$$\varphi \longrightarrow M' \longrightarrow \widehat{G}$$

and to each element in $\Pi_\varphi(M)$ we obtain a subset of $\Pi_\varphi(G)$. Then the sum

$$\sum_{c \in C(G)} \eta(c)\Theta_{c\pi}.$$

may be written as

$$\sum_{c \in C(M)} \eta(c)\Theta_{{\rm Ind}\, c\tau^0}$$

Thus we can expect the character relation in this case to be deduced from a character relation on M

Suppose we have a factorization

$$\widehat{H} \longrightarrow \widehat{M} \longrightarrow \widehat{G}$$

Then we may take $T_G \subseteq M$ and I think that **[17]** if we set $T_G = T_M$ we can prove without difficulty along the lines of Diana's thesis that

$$\mathfrak{W}(T_H, T_G) = \mathfrak{W}(T_H, T_M)\Omega\big(T_G(F), G(F)\big)$$

This is what is needed to lift the character relations.

Thus the following must be proved:

If we have a factorization

$$\widehat{H} \longrightarrow \widehat{M} \longrightarrow \widehat{G}$$

then η is trivial on the kernel of

$$M'(F) \to C(M)$$

This would allow us to reduce to the case that there was no non-trivial factorization of this sort. In other words to the case that 1 is not an eigenvalue of σ on $\widehat{L}(T_{\rm ad}) \otimes \mathbf{R}$ so that the norm of every root $\widehat{\alpha}$ is 0.

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