November 9, 1962 Princeton, N.J.

Dear Professor Harish-Chandra,

Thank you very much for your kindness in allowing me to read your paper on spherical functions. I have waited until now to return it because I wanted to send you my paper on automorphic forms at the same time. If you don't mind I would like to ask you question.

Suppose G is semi-simple (with finite center for simplicity) and K a maximal compact subgroup. Let  $\sigma(K)$  be a matrix representation of K and  $F(g_1, q_2)$  a function on  $G \times G$  such that  $F(g_1q, q_2g) = F(g_1, g_2)$  and  $F(k_1g_1, k_2g_2) = \sigma(k_1)F(g_1, q_2) = \sigma(k_1)F(g_1, q_2)\sigma^{-1}(k_2)$ . In order to apply the Selberg trace formula to concrete problems one has to be able to express

$$\phi_F(\gamma) = \int_{G \setminus G_{\gamma}} \operatorname{tr} \left( F(g\gamma g^{-1}, 1) \right) ds_{\gamma}$$

in terms of the "Fourier transform" of F.  $\gamma = \exp(H)$  is semi-simple and  $G_{\gamma}$  is the centralizer of  $\gamma$  in G. If  $\pi$  is a unitary representation of G let  $\{x_{\ell}^k\}$ ,  $1 \leq \ell \leq d(\sigma)$ ,  $1 \leq k \leq d(\pi, \sigma)$  be vectors such that, for each k,  $(x_1^k, \ldots, x_{d(\sigma)}^k)$  transform under  $\pi(k)$  according to  $\sigma$ . Thus

$$\int_G \sum_{\ell=1}^{d(\sigma)} \overline{F}_{i\ell}(1,g) \pi^*(g) x_\ell^k \, dg = \sum_j \overline{\pi}_{jk}(F) x_i^j.$$

 $(\pi_{ik}(F))$  is the Fourier transform of F at  $\pi$ . If there is a Plancherel formula then

$$F_{pq}(g,1) = \int \operatorname{tr}(\pi(k)\omega_{pq}(g,\pi,\sigma)) \, d\omega(\pi).$$

where the  $(\omega_{pq}(g, \pi, \sigma))$  are a kind of elementary spherical function. Roughly speaking,  $\phi_F(\gamma)$  is a distribution on the functions  $\pi(F)$ . ( $\pi$  varying over the representations occurring in the Plancherel formula.) It will be necessary to prove that this distribution is a function so that

$$\phi_F(\gamma) = \int \operatorname{tr}(\pi(F)T(\pi,\gamma)) d\mu(\pi).$$

where for convenience  $\omega$  is replaced by a measure  $\mu$  simpler than the Plancherel measure. For the applications I have in mind  $\pi(F)$  can be given explicitly so the problem is to determine  $T(\pi, \gamma)$  explicitly. For the discrete series you described in 1956 this is essentially the problem discussed in the paper I am sending you. I have an idea, which I will describe in a moment, for calculating it for the various continuous series. However there is no point in my trying to carry this through if, as is quite possible, you already know what  $T(\pi, \gamma)$  is. I would appreciate it if, when I come to Columbia, you could tell me whether or not this is so.

It is enough to calculate  $\phi_F(\gamma)$  when  $\gamma$  is regular so that  $G_{\gamma}$  can be replaced by the connected component of the identity in the centralizer B of a Cartan subalgebra j. Also

 $\phi_F(\gamma)$  can be replaced by

$$\psi_F(\gamma) = \prod_{\alpha \in P} \left( e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)} \right) \phi_F(\gamma).$$

If p is a polynomial on j invariant under the Weyl group then

 $\partial(p)\psi_F(\gamma) = \phi_{Z_pF}(\gamma)$  (formula for semi-simple groups).

But  $Z_p f(g, 1) = \int \operatorname{tr} (\phi(K) Z_p \omega(g, \pi, \sigma)) d\mu(\pi)$ . For each  $\pi$  there is  $H_{\pi}$  so that  $Z_p \omega(g, \pi, \sigma) = p(H_{\pi}) \omega(g, \pi, \sigma)$ .

Consequently

$$\partial(p)T(\pi,\gamma) = p(H_{\pi})T(\pi,\gamma)$$

and

$$T(\pi,\gamma) = \sum_{s \in W} c(s,\pi) e^{\langle sH_\pi,H\rangle}$$

It should be enough to determine  $c(s,\pi)$  when B is compact. There are a number of boundary conditions that  $c(s,\pi)$  must satisfy. These are probably no use for the discrete series but then  $T(\pi,\gamma)$  is closely related to the trace. It is to be hoped however that for the principal series they will determine  $c(\pi,\gamma)$  Then one will have to combine the two extremes to obtain  $c(\pi,\gamma)$  for the other representations. The boundary conditions that I can see at present are

- (i)  $T(\pi, \gamma)$  is skew-symmetric under the Weyl group of  $K_c$  if  $j \subseteq k$ .
- (ii) According to note at the end of the last 1957 paper in the American Journal  $c_1 = \cdots = c_r$ . The value of the Plancherel measure itself would give another boundary condition but it is to be hoped that it is not necessary to use this.
- (iii) For brevity I will be a bit careless in describing this condition. If  $G_0 = SL(2, \mathbf{R})$  and  $B_0 = \{u_\theta\}, B_{1,0} = \{h_t\}$  in the notation of your note on the Plancherel formula then

$$\lim_{t \to 0} \sinh t \int_{G_0 \setminus B_{1,0}} f(g_0 h_t g_0^{-1}) = c \lim_{\theta \to 0} \sin \theta \left\{ \int_{G \setminus B_0} \left\{ f(g u_\theta g^{-1}) + f(g u_{-\theta} g^{-1}) \right\} \right\}$$

Suppose  $\gamma \in B$  is semi regular and is contained in a non-compact  $B_1$ . Then there is a  $G_0$  (or a covering group of  $G_0$ ) contained in the centralizer of  $\gamma$ . Choose  $\gamma' \in G_0 \cap B$  so that  $\gamma\gamma'$  is regular and  $\gamma'' \in G_0 \cap B_1$  so that  $\gamma\gamma''$  is regular; then

(a) 
$$\int_{G\setminus B} \operatorname{tr} \{ F(g\gamma\gamma''g^{-1}, 1) + F(g\gamma\gamma'^{-1}g^{-1}, 1) \}$$
  
=  $c \int_{G\setminus G_{\gamma}} ds_{\gamma} \int_{G\setminus B_{0}} \operatorname{tr} \{ F(g\gamma g_{0}\gamma' g_{0}^{-1}g^{-1}, 1) + F(g\gamma g_{0}\gamma'^{-1}g_{0}^{-1}g^{-1}, 1) \}$   
(b)  $\int_{G\setminus B_{1}} \operatorname{tr} \{ F(g\gamma\gamma''g^{-1}, 1) \} = c \int_{G\setminus G_{\gamma}} ds_{\gamma} \int_{G\setminus B_{1,0}} \operatorname{tr} \{ F(g\gamma g_{0}\gamma''g_{0}^{-1}g^{-1}, 1) \}$ 

Multiply (a) and (b) by the appropriate factors and send  $\gamma'$ ,  $\gamma''$  to 1. Then the right hand sides will differ only by a constant. If we use induction on the dimension of j the left hand side of (a) can be assumed known in terms of  $\pi(F)$  and the left-hand side of (b) is expressed in terms of  $c(s, \pi)$ . The simplest example is  $SL(2, \mathbf{R})/\{\pm 1\}$ . Then if  $d\mu(\pi) = d\lambda$ ,  $\pi = z_{\lambda}^+$ .  $\sin\theta/2 \ T(z_{\lambda}^+, u_{\theta}) = c_1 e^{\lambda\theta} + c_2 e^{-\lambda\theta}$ 

The first condition gives nothing. The second and third give

$$\lambda(c_1 - c_2) = -\lambda(c_1 e^{2\pi\lambda} - c_2 e^{-2\pi\lambda}); \quad e_1(1 + e^{2\pi\lambda}) + c_2(1 + e^{-2\pi\lambda}) = \alpha.$$

 $\alpha$  is a constant. Solving we obtain  $c_1 = \frac{e^{-\pi\lambda}}{4\cosh\pi\lambda}\alpha$ ,  $c_2 = \frac{e^{\pi\lambda}}{4\cosh\pi\lambda}\alpha$  so that the Plancherel measure is  $\beta\lambda \frac{\sinh\pi\lambda}{\cosh\pi\lambda}$ . These values of  $c_1$  and  $c_2$  agree with the formula in Selberg's paper. I have worked out other examples but at present have no general way of utilizing these

boundary conditions.

Yours truly,

R. P. Langlands

Compiled on April 18, 2025.